Int. J. Solids Structures Vol. 15, pp. 183–191
0020-7683/79/0301-0183/$02.00/0

SOME DYNAMICAL PROBLEMS OF THERMOELASTIC DIELECTRICS†

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(*Received* 19 *April* 1978; *received for publication* 5 *September* 1978)

Abstract—This paper presents a reciprocity theorem for dynamic problems in thermoelastic dielectrics and a derivation of fundamental solutions to the wave equations for infinite continuous media. In particular, Green's functions are obtained in closed form for the displacement and polarization vectors, the temperature and Maxwell's electric fields resulting from various causes, all of which are assumed to vary harmonically with time. The last section is devoted to some consequences of the reciprocity theorem given here.

NOTATION

$\langle\hat{A}\rangle_\alpha$ constants (Appendix B)
b_0 constitutive constant
$b_{\alpha\beta}$ constitutive constants
$\langle\hat{C}\rangle_\alpha$ constants (Appendix B)
$c_{\alpha\beta}$ constitutive constants
$\langle\hat{D}\rangle_\alpha$ constants (Appendix B)
$d_{\alpha\beta}$ constitutive constants
$\bar{E}^{MS}$ Maxwell self-field vector
${}_L\bar{E}$ local electric vector
$\bar{E}^0$ electric force vector
$\bar{f}$ body force vector per unit mass
$\bar{H}$ vector function used in Helmholtz formulae (eqn 3.1)
$\langle\hat{I}\rangle_\alpha$ function defined in eqns (3.19) and (4.5)
$\bar{K}$ vector function used in Helmholtz formulae (eqn 3.1)
$\bar{k}$ surface traction vector
$\mathscr{L}$ Laplace transform
$\bar{n}$ unit outward normal vector
$\bar{P}$ polarization vector
r heat source
r_0 νr
$\bar{S} = \bar{S}^{(0)} + \bar{S}^{(1)}$ surface electric force vector per unit area
S_{ij} components of strain tensor
T absolute temperature
T_{ij} components of stress tensor
t time
$\bar{u}$ displacement vector
v scalar function used in Helmholtz formulae (eqn 3.1)
$\bar{x}, \bar{x}'$ position vectors
β surface charge
$\bar{\Gamma}$ vector function used in Helmholtz formulae (eqn 3.1)
γ scalar function used in Helmholtz formulae (eqn 3.1)
δ_{ij} Kronecker delta
ϵ_{ij} components of electric tensor
ϵ_0 permittivity of vacuum
η entropy density per unit mass
θ incremental temperature
θ_0 positive reference temperature
κ constant (eqn 2.19)
$\bar{\Lambda}$ vector function used in Helmholtz formulae (eqn 3.1)
λ_0 coefficient of thermal conductivity
$\mu, \langle\hat{\mu}\rangle_n$ constants (Appendix A)
ν constant (eqn 2.19)

†Results presented here were obtained in the course of research sponsored by the National Research Council of Canada, Grant No. A-2736.
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ξ constitutive constant
$\hat{\xi}$ $\nu\xi$
ρ mass density
ρ_c charge density
σ constitutive constant
$\hat{\sigma}$ $\nu\sigma$
$^{(\hat{\sigma})}_1$ constants (Appendix A)
τ time
ϕ Maxwell potential
χ scalar function used in Helmholtz formulae (eqn 3.1)
ψ scalar function used in Helmholtz formulae (eqn 3.1)

1. INTRODUCTION

In Toupin's work[1] on elastic dielectrics there is no coupling between the mechanical displacement and the polarization gradient for centrosymmetric materials. This theory was extended by Mindlin[2] and Suhubi[3] by assuming dependence of the stored energy function also on the polarization gradient. Mindlin's theory has been of increasing interest in recent years for the solution of plane wave problems. For example, by including magnetic effects and assuming periodic wave characteristics, the plane wave problem for alpha quartz was investigated in [4]. A linear theory of thermoelastic dielectrics is presented in [5] based on the energy balance, while a complete nonlinear theory for such materials is derived in [6] using the first and second law of thermodynamics and certain invariance requirements.

Betti's reciprocity theorem was generalized to thermoelasticity by Maysel[7] while for the static elastic dielectric case it was recently derived in [8]. The problem of monochromatic three-dimensional waves for classical thermoelasticity has been investigated by a number of authors. In particular, Green functions for such waves in micropolar thermoelasticity were studied by Nowacki[9].

The purpose of this paper is: (i) to derive the reciprocity relation for dynamic problems in thermoelastic dielectrics; (ii) to derive fundamental solutions to the wave equations for the unbounded continuum; in particular, present, in a closed form, Green's functions for the displacement and polarization vectors, the temperature and Maxwell's electric fields resulting from the action of concentrated body force, electric force, charge density and heat supply functions, all varying harmonically with time; and (iii) indicate some consequences of the reciprocity theorem for monochromatic problems in thermoelastic dielectrics.

All symbols are defined where first used in the text and are summarized for convenience in the Notation.

2. THE RECIPROCITY THEOREM

The system of basic linear equations for a thermoelastic dielectric including polarization gradient effects occupying the domain V and bounded by the surface S are given by[5]

(i) *The field equations*

$$T_{ij,i} + \rho f_j = \rho \ddot{u}_j; \quad T_{ij} = T_{ji} \tag{2.1}$$

$$\epsilon_{ij,i} + {}_LE_j + E_j^{MS} = -E_j^0 \tag{2.2}$$

$$-\epsilon_0 \phi_{,ii} + P_{i,i} = -\rho_c \quad \text{in } V \tag{2.3}$$

$$\phi_{,ii} = 0 \quad \text{in } V^* \tag{2.4}$$

$$\theta_0 \dot{\eta} = \lambda_0 \theta_{,ii} + r. \tag{2.5}$$

(ii) *The boundary conditions*

$$T_{ij} n_i = k_j; \quad \epsilon_{ij}^{(1)} n_i = S_j^{(1)}; \quad n_j b_0 = S_j^{(0)} \tag{2.6}$$

$$n_i [P_i - \epsilon_0 \| \phi_{,i} \|] = \beta. \tag{2.7}$$

(iii) *The constitutive relations*

$$T_{ij} = d_{12}\delta_{ij}P_{n,n} + d_{44}(P_{j,i} + P_{i,j}) + c_{12}\delta_{ij}S_{kk} + 2c_{44}S_{ij} + \sigma\delta_{ij}\theta \tag{2.8}$$

$$-{}_LE_k = aP_k \tag{2.9}$$

$$\epsilon_{ij}^{(1)} = b_{12}\delta_{ij}P_{n,n} + b_{44}(P_{i,j} + P_{j,i}) + d_{12}\delta_{ij}S_{kk} + 2d_{44}S_{ij} + \xi\delta_{ij}\theta \tag{2.10}$$

$$\epsilon_{ij} = \epsilon_{ij}^{(1)} + b_0\delta_{ij} \tag{2.11}$$

$$\eta = \sigma S_{kk} + \xi P_{k,k} + c_\epsilon\theta_0^{-1}\theta \tag{2.12}$$

and the kinematic relations

$$E_i^{MS} = -\phi_{,i}; \quad S_{ij} = 1/2(u_{i,j} + u_{j,i}) \tag{2.13}$$

where T_{ij}, ϵ_{ij} and S_{ij} designate the components of the stress tensor, electric tensor and strain tensor, respectively; u_i, P_i, ${}_LE_i$, E_i^{MS}, f_i, n_i are components of the displacement vector, polarization vector, local electric vector, Maxwell self-field vector, body force vector and the exterior unit normal vector, respectively; ϕ, $\|\phi_{,i}\|$, ρ_c, η, denote Maxwell potential, jump in ϕ_i across S, charge density and entropy density, respectively; k_i, $S_i = S_i^0 + S_i^{(1)}$, β and r are surface traction, surface electric force, surface charge and heat source, respectively, while V^* stands for outer vacuum and ϵ_0 its permittivity; θ is an incremental temperature defined by[10, 11]

$$\theta = T - \theta_0 \tag{2.14}$$

where T is the absolute temperature and θ_0 denotes a constant, positive, reference temperature; a, $b_{\alpha\beta}$, $c_{\alpha\beta}$, $d_{\alpha\beta}$, ξ and σ are material constants, c_ϵ is the specific heat and λ_0 denotes the coefficient of thermal conductivity.

Substituting eqns (2.8)–(2.12) into (2.1)–(2.5) yields Navier's equations of thermodielectrics as

$$c_{44}\nabla^2\bar{u} + (c_{12} + c_{22})\bar{\nabla}\bar{\nabla}\cdot\bar{u} + d_{44}\nabla^2\bar{P} + (d_{12} + d_{44})\bar{\nabla}\bar{\nabla}\cdot\bar{P} + \sigma\bar{\nabla}\theta + \rho\bar{f} = \rho\ddot{\bar{u}} \tag{2.15}$$

$$d_{44}\nabla^2\bar{u} + (d_{12} + d_{44})\bar{\nabla}\bar{\nabla}\cdot\bar{u} + (b_{44} + b_{77})\nabla^2\bar{P} + (b_{12} + b_{44} - b_{77})\bar{\nabla}\bar{\nabla}\cdot\bar{P} + a\bar{P} - \bar{\nabla}\phi + \xi\bar{\nabla}\theta + \bar{E}^0 = 0 \tag{2.16}$$

$$\bar{\nabla}\cdot\bar{P} - \epsilon_0\nabla^2\phi = -\rho_c \tag{2.17}$$

$$\left[\nabla^2 - \frac{1}{\kappa}\frac{\partial}{\partial t}\right]\theta - \hat{\sigma}\bar{\nabla}\cdot\dot{\bar{u}} - \hat{\xi}\bar{\nabla}\cdot\dot{\bar{P}} = -r_0 \tag{2.18}$$

where

$$\kappa = \frac{\lambda_0}{c_\epsilon}; \quad r_0 = \nu r; \quad \hat{\sigma} = \nu\sigma; \quad \hat{\xi} = \nu\xi; \quad \nu = \frac{\theta_0}{\lambda_0}. \tag{2.19}$$

Let the dielectric be subjected to two different loading systems, namely $I = \{\bar{f}, \bar{E}^0, \rho_c, \bar{k}, \bar{S}, \beta, r\}$ and $I' = \{\bar{f}', \bar{E}^{0\prime}, \rho_c', \bar{k}', \bar{S}', \beta', r'\}$. Let the two corresponding configurations be defined by $\{\bar{u}, \bar{P}, \phi, \theta\}$ and $\{\bar{u}', \bar{P}', \phi', \theta'\}$. Using Laplace's transform[12] on the basic equations (2.1)–(2.4), (2.6)–(2.11), one obtains

$$\rho\int_V(\tilde{f}_i\tilde{u}_i' - \tilde{f}_i'\tilde{u}_i)\,dV + \int_V(\tilde{E}_i^0\tilde{P}_i' - \tilde{E}_i^{0\prime}\tilde{P}_i)\,dV + \int_V(\tilde{\rho}_c\tilde{\phi}' - \tilde{\rho}_c'\tilde{\phi})\,dV$$
$$+ \int_A(\tilde{k}_i\tilde{u}_i' - \tilde{k}_i'\tilde{u}_i)\,dA + \int_A(\tilde{S}_i^{(1)}\tilde{P}_i' - \tilde{S}_i^{(1)\prime}\tilde{P}_i)\,dA + \int_A(\tilde{\beta}\tilde{\phi}' - \tilde{\beta}'\tilde{\phi})\,dA$$
$$= \int_V[\sigma(\tilde{\theta}\tilde{S}_{ii}' - \tilde{\theta}'\tilde{S}_{ii}) + \xi(\tilde{\theta}\tilde{P}_{i,i}' - \tilde{\theta}'\tilde{P}_{i,i})]\,dV \tag{2.20}$$

where

$$\bar{f}_i(\bar{x}, p) = \mathscr{L}[f_i(\bar{x}, t)] = \int_0^\infty f_i(\bar{x}, t)\, e^{-pt}\, dt; \quad \text{etc.}$$

Next we use the heat conduction equation (2.5), togehter with eqn (2.12) and apply Laplace's transform to obtain

$$p \int_V [\hat{\sigma}(\bar{u}_{i,i}\bar{\theta}' - \bar{u}'_{i,i}\bar{\theta}) + \hat{\xi}(\bar{P}_{i,i}\bar{\theta}' - \bar{P}'_{i,i}\bar{\theta}]\, dV = \int_V (\bar{r}_0\bar{\theta}' - \bar{r}'_0\bar{\theta})\, dV + \int_A (\bar{\theta}_{,i}\bar{\theta}' - \bar{\theta}'_{,i}\bar{\theta})\, dA. \tag{2.21}$$

Substituting eqn (2.21) into (2.20) and using the convolution theorem we obtain the general form of the reciprocity theorem as

$$\begin{aligned}
\nu \int_0^t \Bigg\{ \int_V \Bigg[& \rho f_i(\bar{x}, t-\tau) \frac{\partial u'_i(\bar{x}, \tau)}{\partial \tau} - \rho f'_i(\bar{x}, t-\tau) \frac{\partial u_i(\bar{x}, \tau)}{\partial \tau} + E_i^0(\bar{x}, t-\tau) \frac{\partial P'_i(\bar{x}, \tau)}{\partial \tau} \\
& - E_i^{0\prime}(\bar{x}, t-\tau) \frac{\partial P_i(\bar{x}, \tau)}{\partial \tau} + \rho_c(\bar{x}, t-\tau) \frac{\partial \phi'(\bar{x}, \tau)}{\partial \tau} - \rho'_c(\bar{x}, t-\tau) \frac{\partial \phi(\bar{x}, \tau)}{\partial \tau} \Bigg] dV \\
+ \int_A \Bigg[& k_i(\bar{x}, t-\tau) \frac{\partial u'_i(\bar{x}, \tau)}{\partial \tau} - k'_i(\bar{x}, t-\tau) \frac{\partial u_i(\bar{x}, \tau)}{\partial \tau} + S_i^{(1)}(\bar{x}, t-\tau) \frac{\partial P'_i(\bar{x}, \tau)}{\partial \tau} \\
& - S_i^{(1)\prime}(\bar{x}, t-\tau) \frac{\partial P_i(\bar{x}, \tau)}{\partial \tau} + \beta(\bar{x}, t-\tau) \frac{\partial \phi'(\bar{x}, \tau)}{\partial \tau} - \beta'(\bar{x}, t-\tau) \frac{\partial \phi(\bar{x}, \tau)}{\partial \tau} \Bigg] dA \Bigg\} d\tau \\
= \int_0^t \Bigg\{ \int_V & [r_0(\bar{x}, t-\tau)\theta'(\bar{x}, \tau) - r'_o(\bar{x}, t-\tau)\theta(\bar{x}, \tau)]\, dV \\
& + \int_A [\theta'(\bar{x}, t-\tau)\theta_{,i}(\bar{x}, \tau) - \theta(\bar{x}, t-\tau)\theta'_{,i}(\bar{x}, \tau)]\, dA \Bigg\} d\tau
\end{aligned} \tag{2.22}$$

from which, by assuming all causes and effects to be harmonic in time, we obtain the following simpler form of the reciprocity theorem for an infinite body

$$i\nu \int_V [\rho f_i^* u_i^{*\prime} - \rho f_i^{*\prime} u_i^* + E_i^{*0} P_i^{*\prime} - E_i^{*0\prime} P_i^* + \rho_c^* \phi^{*\prime} + \rho_c^{*\prime} \phi^*]\, dV = \int_V (r_0^* \theta^{*\prime} - r_0^{*\prime} \theta^*)\, dV \tag{2.23}$$

where

$$f_i = f_i^*(\bar{x})\, e^{-i\omega t}; \quad E_i^0 = E_i^{0*}(\bar{x})\, e^{-i\omega t} \quad \text{etc. and } i = \sqrt{-1}.$$

3. FUNDAMENTAL SOLUTIONS FOR INFINITE THERMOELASTIC DIELECTRICS

In the remainder of the paper, discussion is restricted to the case of all causes and effects varying harmonically with time. Using the Helmholtz decomposition theorem for vector fields, one can write

$$\begin{aligned}
\bar{u}^* &= \bar{\nabla}\psi^* + \bar{\nabla} \times \bar{H}^*; \quad \bar{P}^* = \bar{\nabla}\chi^* + \bar{\nabla} \times \bar{K}^*; \\
\bar{f}^* &= \bar{\nabla}v^* + \bar{\nabla} \times \bar{\Gamma}^*; \quad \bar{E}^{0*} = \bar{\nabla}\gamma^* + \bar{\nabla} \times \bar{\Lambda}^*;
\end{aligned} \tag{3.1}$$

which, when used in eqns (2.15)–(2.18), leads to the following two systems

$$(c_{11}\nabla^2 + \rho\omega^2)\psi^* + d_{11}\nabla^2\chi^* + \sigma\theta^* = -\rho v^* \tag{3.2}$$

$$d_{11}\nabla^2\psi^* + (b_{11}\nabla^2 - a)\chi^* + \xi\theta^* = \phi^* = -\gamma^* \tag{3.3}$$

$$\nabla^2\chi^* - \epsilon_0\nabla^2\phi^* = -\rho_c^* \tag{3.4}$$

$$\left(\nabla^2+\frac{i\omega}{\kappa}\right)\theta^*=-r_0^* \tag{3.5}$$

and

$$(c_{44}\nabla^2+\rho\omega^2)\bar{H}^*+d_{44}\nabla^2\bar{K}^*=-\rho\bar{\Gamma}^* \tag{3.6}$$

$$d_{44}\nabla^2\bar{H}^*+[(b_{44}+b_{77})\nabla^2-a]\bar{K}^*=-\bar{\Lambda}^* \tag{3.7}$$

where

$$x_{11}=x_{12}+2x_{44}\quad (x=b,c,d)$$

and where the terms $\hat{\sigma}\bar{\nabla}\cdot\dot{\bar{u}}$ and $\hat{\xi}\bar{\nabla}\cdot\dot{\bar{P}}$ in eqn (2.18) were neglected as "small terms" in deriving these results[13]. The first system represents longitudinal waves and temperature fields. Longitudinal waves can be produced in an infinite medium by a temperature field obeying eqn (3.5) with a given heat source, r_0^*, the sources v^*, γ^*, ρ_c^* and the initial disturbances in ψ^*, χ^*, ϕ^* and θ^*. The transverse waves described by the second system can be produced by the sources $\bar{\Gamma}^*$ and $\bar{\Lambda}^*$ as well as the initial disturbances in $\bar{H}^*$ and $\bar{K}^*$ only.

After some algebra, eqns (3.2)–(3.7) can be recast to read

$$(\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)\psi^*=\sigma_3(\nabla^2+\sigma_4)v^*+\sigma_5\nabla^2\gamma^*+\sigma_6\rho_c^*+\sigma_7(\nabla^2+\sigma_8)\theta^* \tag{3.8}$$

$$\nabla^2(\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)\chi^*=\sigma_9\nabla^4v^*+\sigma_{10}(\nabla^2+\sigma_{11})\nabla^2\gamma^*+\sigma_{12}(\nabla^2+\sigma_{11})\rho_c^*+\sigma_{13}\nabla^2(\nabla^2+\sigma_{14})\theta^* \tag{3.9}$$

$$\nabla^2(\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)\phi^*=-\rho\sigma_6\nabla^4v^*-\sigma_{12}(\nabla^2+\sigma_{11})\nabla^2\gamma^*+\frac{\sigma_{13}}{\epsilon_0}\nabla^2(\nabla^2+\sigma_{14})\theta^*$$
$$+[\nabla^4+\sigma_0(\rho\omega^2b_{11}-ac_{11})\nabla^2-\epsilon_0a\rho\omega^2]\epsilon_0^{-1}\rho_c \tag{3.10}$$

$$(\nabla^2+\mu^2)\theta^*=-r_0^* \tag{3.11}$$

and

$$(\nabla^2+\hat{\mu}_1^2)(\nabla^2+\hat{\mu}_2^2)\bar{H}^*=\hat{\sigma}_3(\nabla^2+\hat{\sigma}_4)\bar{\Gamma}^*+\hat{\sigma}_5\nabla^2\bar{\Lambda}^* \tag{3.12}$$

$$(\nabla^2+\hat{\mu}_1^2)(\nabla^2+\hat{\mu}_2^2)\bar{K}^*=\hat{\sigma}_9\nabla^2\bar{\Gamma}^*+\hat{\sigma}_{10}(\nabla^2+\hat{\sigma}_{11})\bar{\Lambda}^* \tag{3.13}$$

where all new constants introduced are defined in Appendix A.

In order to obtain a solution of the system of eqns (3.8)–(3.13), we assume in succession all source terms, except one, to be equal to zero; (i) $f_i^*\neq 0$; (ii) $E_i^{0*}\neq 0$; (iii) $\rho_c^*\neq 0$; (iv) $r_0^*\neq 0$. For example, for the first case of loading, one finds

$$\psi^*(\bar{x})=\frac{-\sigma_3}{4\pi(\mu_1^2-\mu_2^2)}\int_V v^*(\bar{x}')[(\mu_1^2-\sigma_4)I_1-(\mu_2^2-\sigma_4)I_2]\,\mathrm{d}V(\bar{x}') \tag{3.14}$$

$$\chi^*(\bar{x})=\frac{-\sigma_5\rho}{4\pi(\mu_1^2-\mu_2^2)}\int_V v^*(\bar{x}')[\mu_1^2I_1-\mu_2^2I_2]\,\mathrm{d}V(\bar{x}') \tag{3.15}$$

$$\phi^*(\bar{x}')=\frac{\chi^*(\bar{x})}{\epsilon_0} \tag{3.16}$$

$$\bar{H}^*(\bar{x})=\frac{-\hat{\sigma}_3}{4\pi(\hat{\mu}_1^2-\hat{\mu}_2^2)}\int_V \bar{\Gamma}^*(\bar{x}')[(\hat{\mu}_1^2-\hat{\sigma}_4)\hat{I}_1-(\hat{\mu}_2^2-\hat{\sigma}_4)\hat{I}_2]\,\mathrm{d}V(\bar{x}') \tag{3.17}$$

$$\bar{K}^*(\bar{x})=\frac{-\hat{\sigma}_5\rho}{4\pi(\hat{\mu}_1^2-\hat{\mu}_2^2)}\int_V \bar{\Gamma}^*(\bar{x}')[\hat{\mu}_1^2\hat{I}_1-\hat{\mu}_2^2\hat{I}_2]\,\mathrm{d}V(\bar{x}') \tag{3.18}$$

where

$$(\hat{I})_n=\frac{\mathrm{e}^{i(\hat{\mu})_nR}}{R}\quad (n=1,2);\quad R=[(x_i-x_i')(x_i-x_i')]^{1/2}. \tag{3.19}$$

Analogous expressions, shown in Appendix B, can be obtained for the other three loading cases listed above.

4. GREEN FUNCTIONS FOR THERMOELASTIC DIELECTRICS

Next we determine Green's functions for the displacement, polarization, electric and temperature fields for a thermoelastic dielectric subjected to periodically varying concentrated body force and electric force acting in an arbitrary direction, concentrated charge density and heat source, all applied at a point $\bar{x}'$ of an infinite dielectric medium. The functions v^*, $\bar{\Gamma}^*$, γ^* and $\bar{\Lambda}^*$ are obtained with the aid of the Helmholtz formulae as

$$\{v^*(\bar{x}'), \gamma^*(\bar{x}')\} = -\frac{1}{4\pi}\int_V \{\bar{f}^*(\bar{x}'), \bar{E}^{0*}(\bar{x}')\} \times \frac{\partial}{\partial \bar{x}}[R^{-1}(\bar{x}, \bar{x}')]\,\mathrm{d}V(\bar{x}) \tag{4.1}$$

$$\{\bar{\Gamma}^*(\bar{x}'), \bar{\Lambda}^*(\bar{x}')\} = -\frac{1}{4\pi}\int_V \{\bar{f}^*(\bar{x}'), \bar{E}^{0*}(\bar{x}')\} \times \frac{\partial}{\partial \bar{x}}[R^{-1}(\bar{x}, \bar{x}')]\,\mathrm{d}V(\bar{x}). \tag{4.2}$$

Substituting these two equations into the solutions for the particular loading cases listed in Section 3 (eqns 3.14–3.18) and Appendix B, and using eqns (3.1), one obtains the Green's functions for the following four loading cases:

(i) *Concentrated body force*: $\mathrm{f}_i^{*(j)} = \delta(\bar{\mathrm{x}})\delta_{ij}$

$${}^{f}G_j^{*(i)} = \frac{1}{4\pi}[(\nabla_j\nabla_i - \nabla^2)(\hat{C}_\alpha \hat{I}_\alpha) - \nabla_j\nabla_i(C_\alpha I_\alpha)] \tag{4.3}$$

$${}^{f}P_j^{*(i)} = \frac{1}{4\pi}[\hat{C}_3(\nabla_j\nabla_i - \nabla^2)(\hat{\mu}_1\hat{I}_1 - \hat{\mu}_2\hat{I}_2) + C_3\nabla_j\nabla_i(\mu_1 I_1 - \mu_2 I_2)] \tag{4.4}$$

$${}^{f}\phi^{*(i)} = \frac{1}{4\pi\epsilon_0} C_3\nabla_i(I_1 + I_2) \tag{4.5}$$

where $I_0 = \hat{I}_0 = R^{-1}$ and $(\alpha = 0, 1, 2)$.

(ii) *Concentrated electric force*: $\mathrm{E}_i^{*0(j)} = \delta(\bar{\mathrm{x}})\delta_{ij}$

$${}^{E}G_j^{*(i)} = \frac{1}{4\pi\rho}[\hat{C}_3(\nabla_j\nabla_i - \nabla^2)(\hat{\mu}_1\hat{I}_1 - \hat{\mu}_2\hat{I}_2) + C_3\nabla_j\nabla_i(\mu_1 I_1 - \mu_2 I_2)] \tag{4.6}$$

$${}^{E}P_j^{*(i)} = \frac{1}{4\pi}[(\nabla_j\nabla_i - \nabla^2)(\hat{D}_\alpha \hat{I}_\alpha) - \nabla_j\nabla_i(D_\alpha I_\alpha)] \tag{4.7}$$

$${}^{E}\phi^{*(i)} = \frac{1}{4\pi\epsilon_0}\nabla_i(D_\alpha I_\alpha). \tag{4.8}$$

(iii) *Concentrated electric charge*: $\rho_c^* = \delta(\bar{\mathrm{x}})$

$${}^{\rho_c}G_j^* = \frac{C_3}{4\pi\rho\epsilon_0}\nabla_j(\mu_1{}^2 I_1 - \mu_2{}^2 I_2) \tag{4.9}$$

$${}^{\rho_c}P_j^* = \frac{1}{4\pi\epsilon_0}\nabla_j(D_\alpha I_\alpha) \tag{4.10}$$

$${}^{\rho_c}\phi^* = \frac{1}{4\pi\epsilon_0{}^2}(D_\alpha I_\alpha - I_0). \tag{4.11}$$

(iv) *Concentrated heat source*: $\mathrm{r}_0^* = \delta(\bar{\mathrm{x}})$

$${}^{\theta}G_j^* = \frac{\sigma_7}{4\pi}\nabla_j(A_\alpha I_\alpha) \tag{4.12}$$

$${}^{\theta}P_j^* = \frac{\sigma_{13}}{4\pi}\nabla_j(\hat{A}_\alpha I_\alpha) \tag{4.13}$$

$$^{e}\phi^* = \frac{\sigma_{13}}{4\pi\epsilon_0}(\hat{A}_\alpha I_\alpha) \tag{4.14}$$

where all new constants used in this section are defined in Appendix A. Instead of applying the above concentrated loads at the origin, as was done, one can apply them at a general point $\bar{x}'$ to obtain the Green's functions at point $\bar{x}$, i.e. $G(\bar{x}, \bar{x}')$. Note that in general, μ, ${}^{(}\hat{\mu}\}$ are complex numbers and to obtain a real valued solution for the governing equations, one uses the formulae

$$G(\bar{x}, \bar{x}', t) = Re[G^*(\bar{x}, \bar{x}')\,e^{-i\omega t}]; \quad \text{etc.} \tag{4.15}$$

5. IMPLICATIONS OF RECIPROCITY THEOREM

Assume that at the point $\bar{x}'$ there acts a harmonically varying concentrated force $f_i^{(j)} = \delta(\bar{x} - \bar{x}')\delta_{ij}\,e^{-i\omega t}$ directed along the x_j-axis, and at a second point, $\bar{x}''$, a similar force $f_i^{(k)} = \delta(\bar{x} - \bar{x}'')\delta_{ik}\,e^{-i\omega t}$, directed along the x_k-axis. For this loading system we obtain, from eqn (2.23)

$$\int_V \delta(\bar{x} - \bar{x}')\delta_{ij}\,{}^{f}G_i^{*(k)}(\bar{x}, \bar{x}'')\,\mathrm{d}V = \int_V \delta(\bar{x} - \bar{x}'')\delta_{ik}\,{}^{f}G_i^{*(j)}(\bar{x}, \bar{x}')\,\mathrm{d}V \tag{5.1}$$

from which we conclude that

$${}^{f}G_k^{*(j)}(\bar{x}', \bar{x}'') = {}^{f}G_j^{*(k)}(\bar{x}'', \bar{x}'). \tag{5.2}$$

Similarly, one can show the same symmetry relations for the Green's functions corresponding to the other three concentrated load cases discussed above.

In deriving a second implication, consider that there is a concentrated electric force, ${}^{(i)}E_j^0 = \delta(\bar{x}, \bar{x}')\delta_{ij}\,e^{-i\omega t}$ acting at the point $\bar{x}'$ and a concentrated force, $f_j^{(k)} = \delta(\bar{x}, \bar{x}'')\delta_{jk}\,e^{-i\omega t}$ acting at point $\bar{x}''$. Using the reciprocity theorem, eqn (2.23), one can show that for such a loading condition

$$\int_V \rho\delta(\bar{x} - \bar{x}'')\delta_{jk}\,{}^{E}G_j^{*(i)}(\bar{x}, \bar{x}')\,\mathrm{d}V = \int_V \delta(\bar{x} - \bar{x}')\delta_{ij}\,{}^{f}P_j^{*(k)}(\bar{x}, \bar{x}'')\,\mathrm{d}V \tag{5.3}$$

which yields

$$\rho\,{}^{E}G_k^{*(i)}(\bar{x}'', \bar{x}') = {}^{f}P_i^{*(k)}(\bar{x}', \bar{x}''). \tag{5.4}$$

By analogous reasoning, one can derive for various pairs of concentrated loads the following additional reciprocity relations

$$\rho\,{}^{P_c}G_k^{*}(\bar{x}'', \bar{x}') = {}^{f}\phi^{*(k)}(\bar{x}', \bar{x}''), \tag{5.5}$$

$${}^{E}\phi^{*(k)}(\bar{x}'', \bar{x}') = {}^{P_c}P_k^{*}(\bar{x}', \bar{x}''). \tag{5.6}$$

CONCLUSIONS

The general form of the reciprocity theorem for thermoelastic dielectrics including polarization gradient effects has been derived using the field equations and constitutive relations. Next, the basic solution to the wave equations for an infinite medium has been obtained. In particular, closed form expressions for the wave functions of the temperature and Maxwell's electric fields are formed for an unbounded dielectric subject to the action of various external causes (loads) which are harmonically varying with time. One of the results obtained shows that longitudinal displacement waves are coupled with longitudinal polarization waves in an infinite solid. Secondly, transverse displacement waves and transverse polarization waves are also coupled but do not depend on the temperature or Maxwell's electric field. In the final section, some consequences of the reciprocity theorem have been discussed.

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APPENDIX A

The various constants introduced in the body of the paper are defined as follows:

$$\left.\begin{matrix}(\hat\mu_1)^2\\(\hat\mu_2)^2\end{matrix}\right\} = \frac{(\hat\sigma_1) \pm \sqrt{(\hat\sigma_1)^2 + 4(\hat\sigma_2)}}{2} \qquad \mu^2 = \frac{i\omega}{\kappa}$$

$\sigma_0 = (b_{11}c_{11} - d_{11}^2)^{-1}$; $\quad\sigma_1 = \sigma_0[b_{11}\omega^2\rho - c_{11}(a + \epsilon_0^{-1})]$;

$\sigma_2 = (a + \epsilon_0^{-1})\omega^2\rho\sigma_0$; $\quad\sigma_3 = -\sigma_0 b_{11}\rho$;

$\sigma_4 = -(a + \epsilon_0^{-1})b_{11}^{-1}$; $\quad\sigma_5 = d_{11}\sigma_0$;

$\sigma_6 = -\sigma_5\epsilon_0^{-1}$; $\quad\sigma_7 = \sigma_0(\xi d_{11} - \sigma b_{11})$;

$\sigma_8 = (a + \epsilon_0^{-1})\sigma\sigma_0\sigma_7^{-1}$; $\quad\sigma_9 = \rho\sigma_5$;

$\sigma_{10} = -c_{11}\sigma_0$; $\quad\sigma_{11} = \omega^2\rho c_{11}^{-1}$;

$\sigma_{12} = -\epsilon_0^{-1}\sigma_{10}$; $\quad\sigma_{13} = \sigma_0(\sigma d_{11} - \xi c_{11})$;

$\hat\sigma_8 = -\xi\rho\omega^2\sigma_0\sigma_{11}^{-1}$; $\quad\hat\sigma_0 = [c_{44}(b_{44} + b_{77}) - d_{44}^2]^{-1}$;

$\hat\sigma_1 = \hat\sigma_0[(b_{44} + b_{77})\rho\omega^2 - ac_{44}]$; $\quad\hat\sigma_2 = a\rho\omega^2\hat\sigma_0$;

$\hat\sigma_3 = -(b_{44} + b_{77})\rho\hat\sigma_0$; $\quad\hat\sigma_4 = -a(b_{44} + b_{77})^{-1}$;

$\hat\sigma_5 = d_{44}\hat\sigma_0$; $\quad\hat\sigma_9 = \rho\hat\sigma_5$;

$\hat\sigma_{10} = -c_{44}\hat\sigma_0$; $\quad\hat\sigma_{11} = \rho\omega^2 c_{44}^{-1}$.

$$\hat A_1 = \frac{\hat\sigma_8 - \mu_1^2}{(\mu_2^2 - \mu_1^2)(\mu^2 - \mu_1^2)}; \qquad \hat A_2 = \frac{\hat\sigma_8 - \mu_2^2}{(\mu_1^2 - \mu_2^2)(\mu^2 - \mu_2^2)}$$

$$\hat A_3 = \frac{\hat\sigma_8 - \mu^2}{(\mu^2 - \mu_1^2)(\mu^2 - \mu_2^2)}; \qquad \hat C_0 = \frac{(\hat\sigma_3)(\hat\sigma_4)}{(\hat\mu_1)^2(\hat\mu_2)^2}$$

$$\hat C_1 = \frac{((\hat\mu_1)^2 - (\hat\sigma_4))(\hat\sigma_3)}{(\hat\mu_1)^2((\hat\mu_1)^2 - (\hat\mu_2)^2)}; \qquad \hat C_2 = \frac{((\hat\mu_2)^2 - (\hat\sigma_4))(\hat\sigma_3)}{(\hat\mu_2)^2((\hat\mu_2)^2 - (\hat\mu_1)^2)}$$

$$\hat C_3 = \frac{(\hat\sigma_5)\rho}{(\hat\mu_1)^2 - (\hat\mu_2)^2}; \qquad \hat D_0 = \frac{(\hat\sigma_{10})(\hat\sigma_{11})}{(\hat\mu_1)^2(\hat\mu_2)^2}$$

$$\hat D_1 = \frac{((\hat\mu_1)^2 - (\hat\sigma_{11})^2)(\hat\sigma_{10})}{(\hat\mu_1)^2((\hat\mu_1)^2 - (\hat\mu_2)^2)}; \qquad \hat D_2 = \frac{((\hat\mu_2)^2 - (\hat\sigma_{11})^2)(\hat\sigma_{10})}{(\hat\mu_2)^2((\hat\mu_2)^2 - (\hat\mu_1)^2)}.$$

APPENDIX B

Fundamental solutions, analogous to eqns (3.14)–(3.18), are obtained for the remaining three loading cases as follows:

(i) *For the case*: $E_i^{*0} \neq 0$

$$\psi^*(x) = -\frac{\sigma_5}{4\pi(\mu_1^2 - \mu_2^2)}\int_V \gamma^*(\bar x)[\mu_1^2 I_1 - \mu_2^2 I_2]\,dV(\bar x') \tag{B1}$$

$$\chi^*(\bar x) = -\frac{\sigma_{10}}{4\pi(\mu_1^2 - \mu_2^2)}\int_V \gamma^*(\bar x)[(\mu_1^2 - \sigma_{11})I_1 - (\mu_2^2 - \sigma_{11})I_2]\,dV(\bar x') \tag{B2}$$

$$\phi^*(\bar x) = \epsilon_0^{-1}\chi^*(\bar x) \tag{B3}$$

$$\bar H^*(\bar x) = -\frac{\hat\sigma_5}{4\pi(\hat\mu_1^2 - \hat\mu_2^2)}\int_V \bar\Lambda^*(\bar x)[\hat\mu_1^2\hat I_1 - \hat\mu_2^2\hat I_2]\,dV(\bar x') \tag{B4}$$

$$\bar{K}^*(\bar{x}) = -\frac{\hat{\sigma}_{10}}{4\pi(\hat{\mu}_1^2-\hat{\mu}_2^2)}\int_V \bar{\Lambda}^*(\bar{x})[(\hat{\mu}_1^2-\hat{\sigma}_{11})\hat{I}_1-(\hat{\mu}_2^2-\hat{\sigma}_{11})\hat{I}_2]\,\mathrm{d}V(\bar{x}'). \tag{B5}$$

(ii) *For the case*: $\rho_c^* \neq 0$

$$\psi^*(\bar{x}) = \frac{\sigma_5\epsilon_0^{-1}}{4\pi(\mu_1^2-\mu_2^2)}\int_V \rho_c^*(\bar{x}')[\mu_1^2 I_1-\mu_2^2 I_2]\,\mathrm{d}V(\bar{x}') \tag{B6}$$

$$\chi^*(\bar{x}) = \frac{1}{4\pi}\int_V \rho_c^*(\bar{x}')[D_\alpha I_\alpha]\,\mathrm{d}V(\bar{x}') \tag{B7}$$

$$\phi^*(\bar{x}) = \frac{\chi^*(\bar{x})-\phi^{0*}(\bar{x})}{\epsilon_0} \tag{B8}$$

where

$$\phi^{0*}(\bar{x}) = \frac{1}{4\pi}\int_V \frac{\rho c(\bar{x}')}{R(\bar{x},\bar{x}')}\,\mathrm{d}V(\bar{x}').$$

(iii) *For the case*: $r_0^* \neq 0$

Assuming θ is a known function, we obtain

$$\psi^*(\bar{x}) = -\frac{\sigma_7}{4\pi(\mu_1^2-\mu_2^2)}\int_V \theta^*(\bar{x}')[(\mu_1^2-\sigma_8)I_1-(\mu_2^2-\sigma_8)I_2]\,\mathrm{d}V(\bar{x}') \tag{B9}$$

$$\chi^*(\bar{x}) = -\frac{\sigma_{10}}{4\pi(\mu_1^2-\mu_2^2)}\int_V \theta^*(\bar{x}')[(\mu_1^2-\sigma_{14})I_1-(\mu_2^2-\sigma_{14})I_2]\,\mathrm{d}V(\bar{x}') \tag{B10}$$

$$\phi^*(\bar{x}) = \frac{\chi^*(\bar{x}')}{\epsilon_0}. \tag{B11}$$

Now, assume that $r_0^*(\bar{x})$, is a known function. Then eqns (3.8)–(3.11) imply

$$(\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)(\nabla^2+\mu^2)\psi^* = -\sigma_7(\nabla^2+\sigma_8)r_0^* \tag{B12}$$

$$(\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)(\nabla^2+\mu^2)\chi^* = -\sigma_{13}(\nabla^2+\sigma_{14})r_0^* \tag{B13}$$

$$(\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)(\nabla^2+\mu^2)\phi^* = -\epsilon_0^{-1}\sigma_{13}(\nabla^2+\sigma_{14})r_0^* \tag{B14}$$

and the solutions to these equations take the form

$$\psi^*(\bar{x}) = \frac{\sigma_7}{4\pi}\int_V r_0(\bar{x}')[A_\beta I_\beta]\,\mathrm{d}V(\bar{x}') \tag{B15}$$

$$\chi^*(\bar{x}) = \frac{\sigma_{13}}{4\pi}\int r_0(\bar{x}')[\hat{A}_\beta I_\beta]\,\mathrm{d}V(\bar{x}') \tag{B16}$$

$$\phi^*(\bar{x}) = \frac{\chi^*(\bar{x})}{\epsilon_0} \qquad \beta = 1, 2, 3 \tag{B17}$$

where

$$I_3 = \frac{e^{i\mu R}}{R}.$$